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First Year University Students` Use of Words, Symbols and Images to Convey Mathematical Ideas: A Case of Definitions

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ABSTRACT

This qualitative case study analyzed how first year university mathematics students used words, images, and symbols to convey the same mathematical ideas. The study was located within the interpretivist paradigm and took naturalistic methodology. Twenty-six first year students were purposefully selected to participate in the study. Data was collected through analyzing students' assignments scripts followed by semi-structured interviews. The study sought to answer the questions: How did the first-year university students` use of words, symbols and images to convey mathematical ideas. The data were analyzed using Lave and Wenger`s situated learning and Seo ` mathematical communication theories. Data analysis focused on the structure and usage of symbols, images, and words to communicate mathematical ideas. The result of the study shows that the students experienced difficulties in using words, images, and symbols to communicate the same mathematical idea. There were contradicting meanings between images, words, and symbols usage in attempt to define same concept. It is recommended that encouraging students to wave between words, symbols, and images to communicate mathematical ideas will improve understanding of mathematical concepts. First year university mathematics teachers are encouraged to not only use one method of mathematical communication when defining concepts. Moreover, it is recommended that formal mathematics definition be used after students grasped the meaning of concepts using everyday day words and images. The abstractness should only follow the definitions are understood intuitively and can be represented diagrammatically or with natural language.

KEYWORDS

Mathematical communication; situated learning; community of practice; university students; transition.

INTRODUCTION

Transition from school to university mathematics is a huge problem involving quite a few shifts in ways of doing and practicing mathematics. At school level mathematics is mostly about computations and mastering procedures. However, the university mathematics demand much in-depth understanding and focuses on formal definition of concepts and proving mathematical facts (Wenger, 2000). It is possible to pass school mathematics solely through the use of factual recall, a skill that require little knowledge of mathematics (Collins et al., 2018; Weber, 2001). The transition involves revisiting mathematical objects used in school, defining concepts before using them in proofs, and learning how to use formal mathematical language and coherent use of words, images, and symbol to construct mathematical arguments (Mamolo, 2010). These shifts are many, and research has shown what students experience various challenges in the transition period. Due to conflicting engagement standards in different communities of practice at school and in the university, students' positions imply diverse identity experiences (Wenger, 2000). Mathematical writing and communication skills, institutional changes, social changes, and changes in the subject's substance are all possible outcomes of the transition from high school to university (Alcock & Simpson, 2002). The aim of this study is to analyze how first-year university students use words, images, and symbols to convey mathematical ideas. The study was guided by the question: How did students use words, images and symbols to communicate same mathematical ideas?

LITERATURE REVIEW

Writing can be identified as the single most consequential technology ever invented. The cultural anthropologist Jack Goody (1987) referred to writing as “the technology of the mind” (p.112). Philosopher Plato described writing as a form of communication that can travel in time and space away from its author (Plato, 360BC). According to Plato, the only way a text can answer questions about its meaning is to have the writer interpret it. The writer of the text may also have challenges in recalling the ideas that went into the text during production. Thus, a written text being separated in time and space from its author, must do all communication work and the author must consider all possible readings of different interpreters (Gee, 2015). The need to consider all possible readers while the author of text is not available to answer questions about possible interpretations, make writing very complex, one reason Plato preferred dialogue instead. The possible reading of text is acknowledged in many theories of learning such as discourse theory and semiotics (Porter & Masingila, 2000). In line with Plato`s view about written texts, students` use of words, symbols, and images to convey mathematical ideas was analyzed not only by examining their written responses to mathematical tasks, but students were also interviewed to get the voice that is absent on the written texts.

Transition from School to University Mathematics Learning

Gueudet (2008) classifies the organization of knowledge, thinking mode, definition of concepts, proofs, didactical transpositions, mathematical writing, and mathematical communication while examining research that concentrate on the transition between high school and university. In their study, De Guzman et al. (1998) found that university mathematics learning activities are more formalized, rigorous, and abstract, and that some concepts are changing status when they move from a school to a university environment. According to De Guzman et al. (1998) and Gueudet (2008), formal definitions and proofs of mathematical concepts are crucial components of university mathematics and the main transition that students are expected to make during their first year at a university.

Writing and learning mathematics

Many publications and journals mention the advantages of combining mathematics and writing (Adams, 2003). The improvement of students' knowledge and the opportunity for teachers to gain insight into their students' thinking are two of the most frequently mentioned pedagogical justifications for employing writing in mathematics classes (Morgan 2001; Martinez & Dominguez 2018). Martinez and Dominguez (2018) claim that when teachers encourage their students to write in mathematics, they can better understand how those students think mathematically, identify any misconceptions they may have, and assess their own teaching methods. Writing is a means of communicating and of developing mathematical understanding as well as a means of negotiating mathematical context and language to create mathematical knowledge (Kuzzle, 2013; Seo, 2009). Researchers contend that writing in mathematics results in improved dialogue between students and teachers, which in turn result in students' omissions and misconceptions communicated more clearly and critical thinking, understanding and problem solving improved (McMillan, 2017; Weinhuber et al., 2019). Writing mathematics is complex since mathematical texts are more conceptually dense than other genres of writing (Seo, 2009) and are replete with symbolic and linguistic conventions which make navigating the text challenging (Adams, 2003).

Theoretical Framework

This study used Lave and Wenger`s (1991) situated learning theory and Seo (2009) mathematical communication Frameworks. At the heart of Lave and Wenger`s (1991) theory of situated learning is the notion that learning is fundamentally social and integrally related to an individual`s involving identity in a community of practice. CoP constitutes three core dimensions: What is it about (its joint enterprise), how it functions (the relation of mutual engagement that minds members together), and what capacity is produced (the shared repertoire of communal resources that members have developed over time) (Lave and Wenger, 1991). Learning is considered as increasing participation in CoP, which concerns the whole person acting in the world (Lave & Wenger, 1991). Mathematics is a human practice, depending crucially on the consensus of the community and is very much a socialization process. As social enterprise, mathematical knowledge is subject to consensual validation and shared meaning by members of the community. This is keeping with the view that Mathematics is an ever-changing field that exists not as abstraction but as a piece of linguistic and cultural fabric (Martinez & Dominguez 2018). The change also includes from one community to another (Morgan, 2001).

Studying mathematics at university level means entering a new community where the practice of being a student differs from that of the secondary school community. The need to shift to new ways of being and belonging signifies the need for developing a new identity of practicing mathematics (Sfard, 2008).

Mathematics is a written language as with all languages, it has specific elements to it. Seo (2009), conceptualize Mathematical writing as thematic condensation of terms, symbols, and images to convey mathematical knowledge and meaning. Thus, mathematical writing is comprised of three elements: symbols, normalizations, and images (Seo, 2009). The intertwining of these three elements makes mathematical communication possible between the students and their audiences. To be fully mathematically literate, students need to be able to understand and negotiate mathematical ideas using all the three elements of mathematical communication. It is the role of the teacher to assist the students to effectively communicate their using all the three elements. Symbols are marks on a surface, and the context of the mark determines its meaning (Rotman, 2000). Nominalizations are mathematically specific terms. The context in which these words are used will determine their precise meaning (Seo, 2009). For example, there is more than one meaning of the word "*rational*. In English class, it can mean orderly or logical thought, while in mathematics, the meaning is of ratio or of a fraction (Stein, 1980). Lastly, there are images. Images comprise of all mathematical writings that are not symbols or nominalization. Images are used to illustrate mathematics knowledge/ situation and /or used as a tool to organize students` mathematical knowledge. Depending on the mathematical writing situation, these images may be used either alone or in conjunction with other ones, The beauty of images is that they can convey different kinds of information in one entity (O`Halloran, 2008). Mostly commonly used images are graphs (Seo, 2019). While the school mathematical writing is characterized by excessive use of mathematical symbols without the use of words and images to unpack the ideas and meaning contained in the symbols, the abstractness of university mathematics demand that the students be able to use both words and symbols or images to produce meaning mathematical arguments (Kuzzle, 2013). Teachers as member of schools CoP have consensual validation on the use of symbols to explain mathematical ideas. However, the use of only symbols is no longer sufficient or validated by university mathematics CoP as acceptable way to communicate the mathematical ideas. Mathematical writing aims to explain to the reader the rationale behind the process used to arrive at the solution. One can only show that they have spent some time making calculations if they have a large quantity of calculations without any context or explanation. The ideas are however missed in a set of calculations without any justifications. The mathematics is the ideas. Hence a page of computational symbols without explanations contains no mathematics (Kuzzle, 2013).

METHODOLOGY

This qualitative case study was located within the interpretivist paradigm and took naturalistic methodology. Purposive sampling was used to select twenty-six (26) first year mathematics students as participants. The 26 participants consisted of 15 females and 11 males aged between 18 and 32 years. Purposive sampling allowed the researcher to select individuals that are likely to yield a better overview of the issues under investigation (Leedy & Ormrod, 2005). Data were collected through analyzing of students' assignments scripts and semi-structured interviews. Firstly, the students` written assignments were examined to see how students use words, images, and symbols to represent mathematical definitions. Then the scripts were categorized with focus on scripts that show contradicting meaning between use of symbols, images and words to explain the same mathematical idea. From the 26 students, five students were interviewed guided by the gaps in their written narratives that shows contradicting meaning from different modes of mathematical communication. To validate the reliability of the interview questionnaires, mathematics teachers and researchers reviewed the questions for clarity and specificity. The researcher was the course instructor. Ethical considerations were observed when gathering and presenting data. Anonymity and confidentiality are assured, and the results of this study are reported using only alpha numeric codes. For example, when referring to Mathematics Student number one, the code MS1 is used. The students were asked to define an injective and surjective function and provide a sketch to support their definitions (See appendix A). Thematic analysis was used to break down the data, and all qualitative data were coded to look for recurring themes and patterns (Adu, 2019).

RESULTS

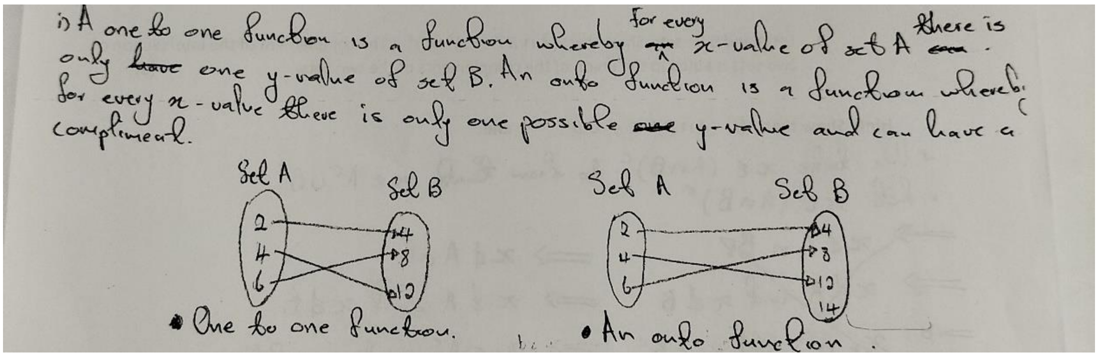
The students were asked to define injective (one to one) and surjective (onto) function using words, symbols and images (See appendix A). A function is injective, or one-to-one, if each element of the codomain is mapped to *at most* one element of the domain (Symbolically: $\forall x, y \in X, f(x) = f(y) \to x = y$). The function is surjective, or onto, if each element of the codomain is mapped to by *at least* one element of the domain (Symbolically: : $\forall y \in Y, \ni x \in X\ s.t\ f(x) = y$. The use of multiple representatives was opportunity for students to demonstrate their understanding of the concepts and for the teacher to diagnose the misconceptions (Martinez & Dominguez 2018). Defining routine is new to first year university mathematics students and as such the question serves as an induction to the practices of university mathematics community (Gueudet, 2008). While students used words, symbols, and images to define the concepts of injective and surjective functions. However, the students' responses displayed contracting meaning between words, symbols and images used to define the same concept.

The student [MS3] combined words and symbols to define a one-to-one function as "a function whereby for every $x-value$ of the set A there is only one $y-value$ of the set B" (figure A). Firstly, the definition presented by [MS3] allows two elements in set A to be mapped to one element in set B (many to one). Many to one is of course a function but not an injective function. However, the student`s understanding of a one-to-one function is made clear by the image presented. The images clearly show a one correspondence between elements of set A and elements of set B, hence a one-to-one function. Moreover, there are contradicting meanings between words used and the image. In defining an onto function, [MS3] writes "An onto function is a function whereby for every $x-value$ there is only one $y-value$ and can have a complement". To start with, the definition of onto-ness has nothing to do with concept of "complement". Again, if we remove the lost word "complement" the student defined the onto function the same way he defined a one-to-one function. Moreover, the definition of onto-ness pays special attention to the elements in the codomain (Set B), namely all elements in the set B need to have at least one element in set A mapped to them. The student said nothing to this extend. On the image used by [MS3] to define onto function, we see that element 14 in set B, is a spectator element (nothing mapped to it from set A), which tells that the function is not onto for an onto function there will be no spectator element in the codomain (set B). When probed during the interview [MS3] said:

It is hard for to formally write the definition of one to one an onto, but I know what it means, I just fail to memorize all the words.

Figure 1

MS3 respond



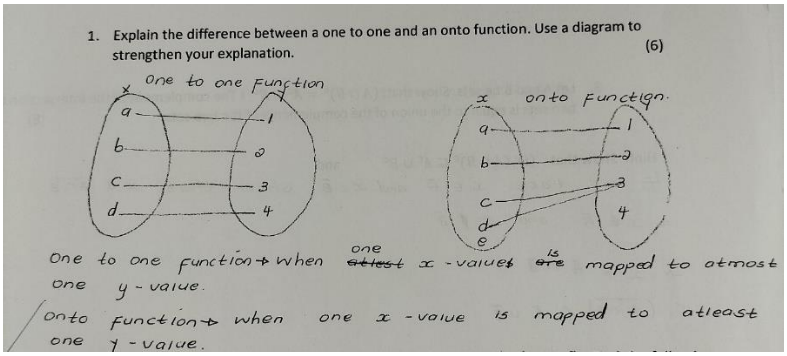
The wording changed slightly for student [MS8] when defining both one to one and an onto function. But [MS8] `s solution in both words and images, share the same commognitive conflicts with [MS3]. The student defined a one-to-one function as "a function when $x-value$ is mapped to at most one $y-value$". This definition does not only allow two elements in the domain to be mapped to same element in the co-domain (many to one) but also allow an element in the domain to be mapped to nothing in the codomain (at most one tells the maximum of one, but no minimum is mentioned). Like [MS3], the student [MS8] diagrammatic representation is correct definition of one to one. When defining an onto function, the student writes a function is onto "when one $x-value$ is mapped to at least one $y-value$". This definition is more problematic as it allows one element in the domain to be mapped to many elements in the codomain (one – many), therefore this relationship does not even define a function. Once the relation failed to be a function, there is no question about its suitability to even be analyzed to check if it is an onto function. An onto function must first be a function. While the words and symbols do not fit the condition for a function, the image presented is that of a function although not an onto as the element 4 in the codomain is a spectator element. We also noticed that the student put an element "e" in the domain as spectator, which is not permissible. During the interview, the student said:

"I know that the number 4 must be mapped to something in input set for the function to be onto, I just forgot to connect arrow from e to 4" [MS8].

Indeed, if the student put the arrow from e to 4 as he says, the sketch will be that of signaling an onto function.

Figure 2

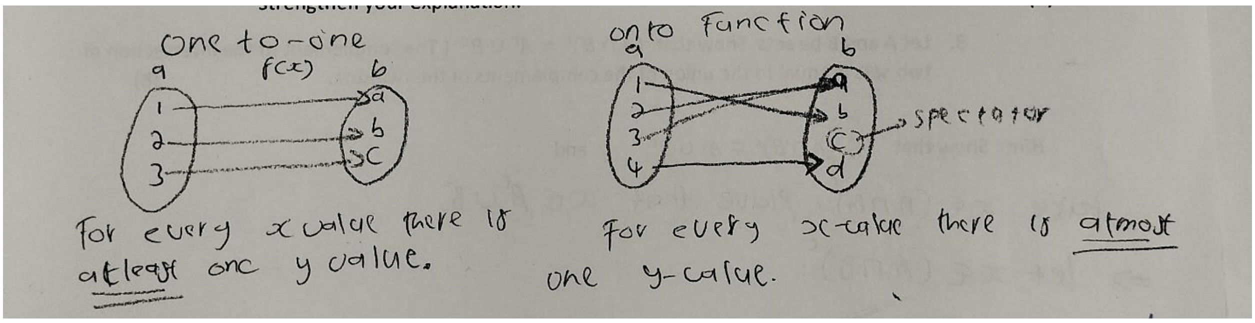
MS8 respond.



The mis match between the meaning contained on word use and images is also seen when students use quantifying words "at least" and "at most". When defining onto function, the student [MS12] says "for every $x-value$ there is at most one $y-value$". The student here clearly did not define an onto but a function for his definition allows many to one and beautifully disqualify one to many. The image pretended in attempt to define onto-ness, show that c is spectator and the student labeled it as such "spectator". But the availability of a spectator element in the codomain is exactly the core factor that makes the function not to be onto, yet the student claims he is defining an onto function and still tells the reader that a spectator element exists. Again, in defining a one-to one function, the student [MS12] writes "for every $x-value$ there is at least one $y-value$. This definition allows one to many and therefore does not even define a function. Again, in this case the diagram is very clear on what one-one mean.

Figure 3

MS12 respond.



We also notice another pattern of contradicting message conveyed by words and the corresponding image. [MS16] define one-one as function where "every x value relates to only one y value ". It was not clear what the student meant by "relate". During the interview with [MS16] to get what he meant by "relate" in his definition he says:

"*By relating I mean a relationship whereby for every element x in one set there is only one y value in the other set, the same relationship is also for y, for every y value there is only one x".*

The respond given during interview is acceptable and clear definition of a one-to-one function. During writing, the student confused relate with mapping. By saying that x relates to y, he is not talking about the mapping but allow any form of relationship. So, the student use mathematical words loosely. For example, if one has a number 3 in set A and number 4 in the set B, then two elements relate because one element is one more than the other. That is a relationship between the two elements, but it does not mean 3 is mapped to 4. So, relating cannot be interchange with mapping.

The student was further asked, if this is what you meant, why didn`t you write exactly what you just said now, [*MS16]* responded:

"I did no think mathematics lecturer will love and appreciate such a long answer and with so many words, I know they like symbols and maybe images". [MS16] further says:
I have always done well in mathematics, I am good with calculations and solving proper maths problem, but this year was very difficult although I passed. I do not like writing explanations

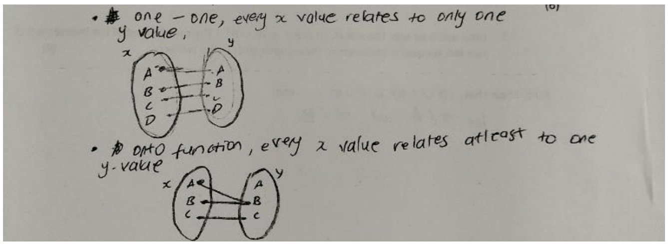
The student [MS16], like many others, did not see explanation of the ideas behind the symbols as mathematics. Mathematics was usually presented to the student in an absolutist approach with fixed algorithms, without any attempt to explain the process and provide a road map (Moloi & Matabane, 2020). To these students, learning mathematics is about equations, formulars and manipulation of symbol. However, a list of calculations without any explanations omits ideas. The ideas are the mathematics. Therefore, students must write both, the calculations, and explanations, to show their complete understanding of the mathematical concept (Kuzzle, 2013).

The image presented by [MS16] demonstrated confusion between how elements in the set and the set themselves are denoted. In Mathematics, upper cases are used to denote sets and lower cases to denote elements in the set. However, the students have labelled the sets using lower case **(**x and y**)** and elements inside the sets by upper cases (A, B, C, D). The interpretation of this is that we have sets inside elements, which is impossible. Again, students mapped set A to set A and set B to set B, another impossible relationship as it maps many elements to many elements.

"I was not aware that the sets must be labelled by capital letters, I thought what is important is to label them as I did. All I know, is that in class the teacher uses small letter as elements and capital letters, but thought was just his choice' [MS16]

Figure 4

MS16 respond.



The students did not see the use of upper cases as the standard way of denoting sets in mathematics community and how the language of mathematics is used. He thought he has choices about communicating mathematics and not focused on the endorsed and acceptable ways of communicating mathematics and in a way gain membership to the community.

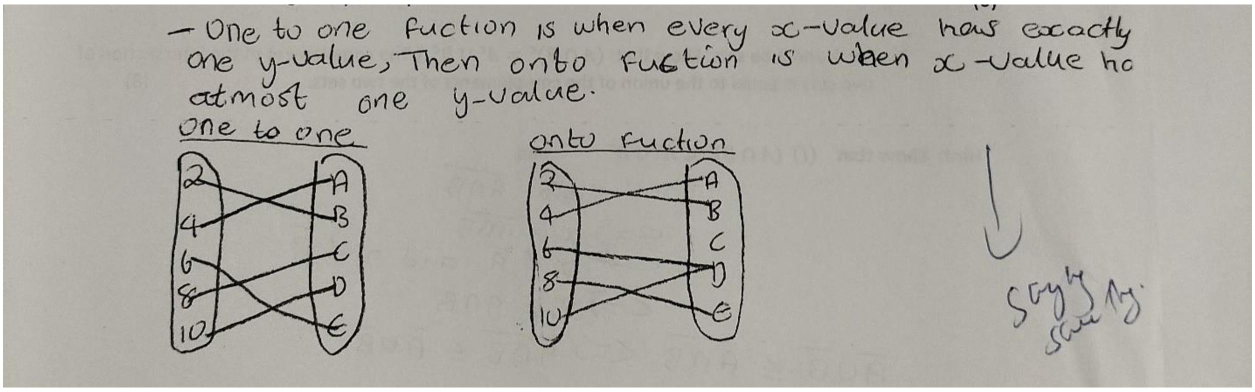
Like [MS18], the students [MS20] put upper cases in the codomain when defining both onto and one-one functions. In the case of [MS20], the sets of domain and codomain are not labelled, and he used numerals in domain and sets in the codomain. According to the student, we can have a numeral 2 mapped to the whole set A, this is impossible unless A is a singleton (as set consisting of only one element). However, in this case, the student nowhere mentioned that the sets A, B, C, D are singletons. During interview, [MS20] says:

"I used A, B.C, D, E as elements and not sets. I did not know that just by using capitals it will be interpreted as sets. This is unfair for alphabets because in the domain I used numbers and with numbers one cannot tell if they are capitals or not. In future, I will use numbers in both sets to avoid trouble".

The student is now trying to find alternatives to defining the concepts and not focusing on the bigger picture of learning the language accepted by mathematics and the meaning that is contained by using capital letter in the context of sets.

Figure 5

MS20 respond



Finally, the student [MS22] start by defining one to one function by saying "is when the two equations are equal to each other for example $x = y$". From his definition, the student confuse mapping as equality. When x is mapped to y it does not mean that they are equal. When asked about the equality, [MS22] says:

"Sir, I did not mean equal as the same, but here I mean what one x is gives one y ".

What student wrote, is different to what she says. Her verbal explanation has logic and is different from the meaning conveyed by her written text. The written text reads " $x = y$ ". The image representation is more aligned to what she said verbally than what she wrote. Indeed, her image shows the part she said during interview "*but here I mean what one x is gives one y ".* We see this because 1 is mapped to an element 'a" and 2 is mapped to an element "b". The part of them being equal as her written text made us conclude is no longer visible.

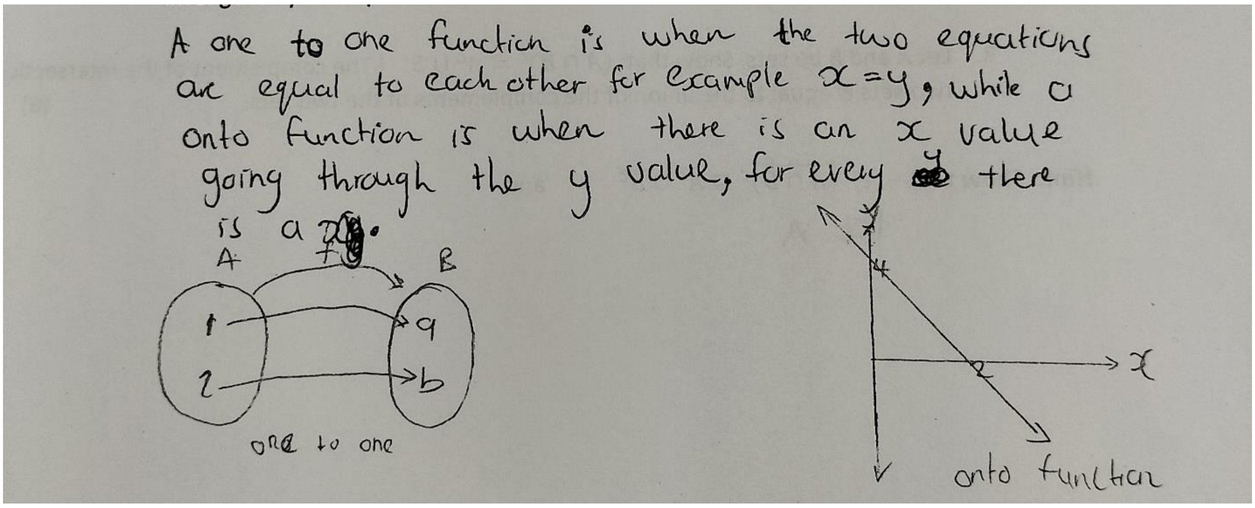
Again, the student [MS22] define an onto function as "when there is an x value going through the y value, for every y there is a x". The meaning of going through is not clear and if the student could remove the first part (when there is an x value going through y value) and only use latter part (for every y there is a x) this is correct and precise definition of onto-ness. Such a precise definition would align very well with her line graph. According to [MS22]:

"This is how my school taught me, by going through, it means y is the images of x ".

Using mathematics language loosely, is common characteristic of school mathematics discourse.

Figure 6

MS22 respond



DISCUSSIONS

The ability to explain mathematical ideas using multiple representations is a decisive test of one`s understanding of the topic (Friesen, 2017). The result of the study shows that students experienced difficulties in representing the same mathematical ideas using different mathematical communication forms. There is contradicting meanings between images, words and symbols while defining the same mathematical concepts. On one hand, some students were able to present formal abstract definitions but unable to put the same definitions in everyday language or using images. The cryptic collections of symbols without attempt to dig meaning behind the symbols is common in schools and hide learning inadequacies and fragility of knowledge (Bardini & Pierce, 2015; Kuzzle, 2013). It is quite possible to memorize the abstract definition having limited understanding of the subject matter, but it will be a challenge to represent the idea in different mathematical writing forms (Tall & Vinner, 1981; Fang & Schleppegrell,2010; Skovsmose, 2013). Even the most capable students were challenged when expected to offer explanation of the ideas behind the symbols.

On the other hand, students found great difficulty in constructing formal definitions of injective and surjective function but gave correct diagrammatic representations. The university mathematics discourse value and reward formal definition using symbols and not images. However, at the school level, images are acceptable ways of showing that a function is injective by using vertical line test and other diagrammatic representations (Guzman, 2002; Romberg & Shafer,2020). While traditional university teaching start with formal definitions and then try to explain the concepts using words and sometimes images, students suggest that it will be easier to understand definitions if images and words are used first, and once students have that diagrammatic understanding, the formal definitions can then be introduced. As the student [MS10] says: " *I have learned a lot by trying to define same concept using the three modes of mathematical communication. Learning to express Math idea without symbols is very interesting and helpful. I found it much easier to move from words and images to symbols, than starting with the formal definition, it is scary and sometimes meaningless forcing us to memorise"*

The comment made by MS10 resonates with the arguments presented by Machaba (2017) when making distinction between mathematics and mathematical literacy (ML). The author argues "ML is also associated with reasoning and problem-solving strategies while mathematics is seen as a discipline that deals primarily with the application of rules" (Machaba, 2017: 106). The blind application of rules without deeper understanding of the meaning behind rules is common in school mathematics discourse.

Conclusion and Recommendation

The school mathematics community value images and graphs as acceptable ways of demonstrating understanding of mathematics concepts. However, the university mathematics community consider the use formal definitions consisting of symbols and words as acceptable ways to define mathematical concepts. The study showed that the ability to wave between words, symbols and images encourage logical thought, accuracy and attention to structure and economy of thought. The study recommend that the university teachers embrace and encourage multiple representations of mathematical thoughts and ideas when defining mathematical concepts. Allowing students to use multiple mathematical communications to support their case teach and help learners to see relevance of mathematics. The skill to communicate mathematical ideas without abstract mathematical symbols is very crucial as it allows one to communicate the mathematical ideas even to audiences outside the discipline of mathematics (non-experts). The study also recommends that university first year mathematics teachers introduce mathematical definitions by first starting with the less formal understanding (images and words) before the formal abstract use of symbols can be introduced to define mathematical concept. Once the students grasp the concepts in less formal way, they will not only better understand the formal definition, but they will actively be taking part in constructing the formal definitions and be transitions better to the university community and university mathematical writing discourse of waving between words, symbols, and images.

Limitations of the study.

The study was a small-scale study from one first-year module at one university in one country. In this study, the main limitation was the use of one set of students, from one teacher, one module, at one university. Therefore, the results of this study cannot be generalizable for all modules or other universities as they are context specific.

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